# SQUARING A CIRCLE AND VICE VERSA – MADE SIMPLE

(FOR ANY DESIRED APPROXIMATION)

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**This paper does a comparative study of the efforts made by Bodhayana of vedic age and Srinivasa Ramanujan of ninteenth century towards the constructional solutions of one of the unsolved problems of antiquity namely 'squaring a circle and vise versa'. It also throws some light on the weaknesses of these constructions, consequent of which a new flexible method is suggested.**

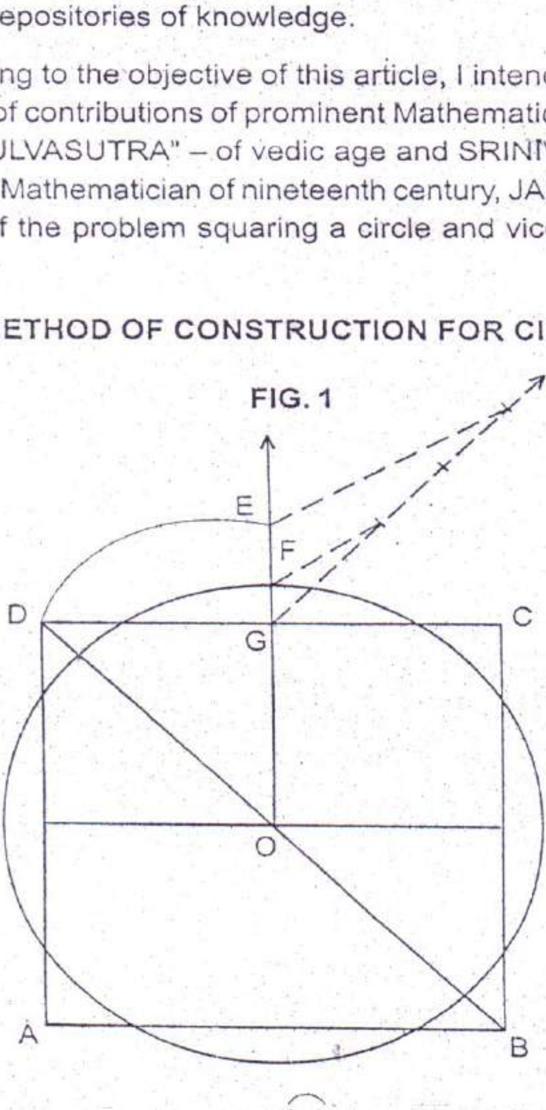
**History :**

A look into the history of Indian Mathematics reveals that, in the days of vedic civilization, Indians felt that it was necessary to perform sacrificial rites to please gods and have boons from them. For this they needed different, specific shapes of altars or 'Vedis' of definite areas so as to make their offerings acceptable to gods. Especially when they had to construct an altar called 'GARHAPATYA' altar, some opined that it should be of square shape while some others asserted that it must have circular shape. Consequently there arose the need for circling a square and vice versa, since in both the cases the area is to remain unaltered. Thus the impossible problem (by virtue of its involvement with the Transcendental Number, $\pi$ ) squaring a circle and vice versa is as old as the Vedas, the repositories of knowledge.

Before proceeding to the objective of this article, I intend to, let us have a glance at the different modes of contributions of prominent Mathematicians, BODHAYANA , the author of the oldest SULVASUTRA" – of vedic age and SRINIVASA RAMANUJAN, the world renowned Indian Mathematician of nineteenth century, JAKOB DE GELDER (1849) towards the solution of the problem squaring a circle and vice versa upto the nearest accuracy.

## BODHAYANA'S METHOD OF CONSTRUCTION FOR CIRCLING A SQUARE

**FIG. 1**



**Construction :**

1. Let ABCD be a square with center O.
2. Turn OD till it assumes the position of OE perpendicular to DC, cutting DC in G.
3. Take GF equal to $\frac{1}{3}$ GE.
4. With OF as radius, describe a circle which will be equal in area to the square ABCD.

On calculation we get $\pi = 3.088 = \frac{AB^2}{OF^2}$

(For proof see Appendix – I)

**BODHAYANA'S CONSTRUCTION FOR SQUARING A CIRCLE :**

Divide the diameter into eight equal parts, and again one of these parts into twenty-nine parts; of these twenty-nine parts and the sixth part of the one left parts less, with the eighth part of the sixth part. [In this case also π equals to 3.088. For proof see Appendix – II]

Since this construction becomes more clumsy, it is not depicted here.

This value is Just better than the value contained in The Mahabaratha and the Holy Bible. (for the relevent versess see appendix IV)

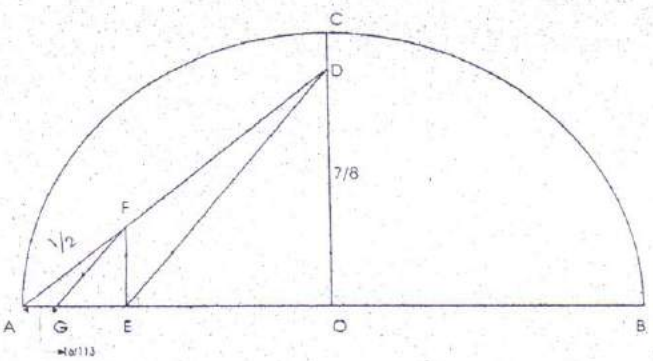
**Remark :**

From these two constructions, it can be assertained that Bodhayana might have adopted different modes of constructions for the same value of π, because either of these two constructions provide iota of possibility for reversal i.e., squaring a circle from circling a square or vice versa.

JAKOB DE GELDER's Construction Of π

In 1849, Jakob de Gelder gave an intersting construction for π using the convergent 355/113 This construction runs as follows.

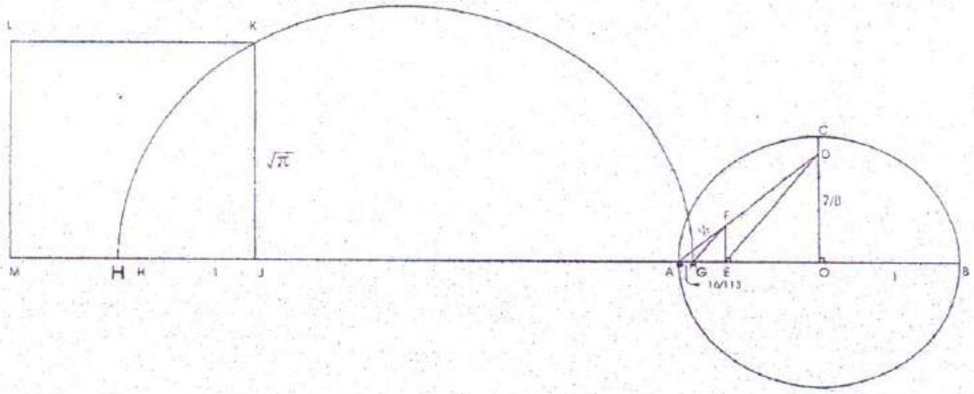
**FIG 2.**




Let O be the centre of a circle with radious OC = 1. Let AB be a diametere perpendicular to OC. Let OD = 7/8 and AF=1/2 : Draw FE parallel to CO and FG parallel to DE. This AG becomes equal to 16/113. Then 3.OC+AG becomes equal to 3.1+16/113 = 355/113.(For poof see Appendix III )

Using this method we can construct a square equal in area to the given circle. The method goes as follows.

FIG. 3




1. Take the circle to be coverted into a square of equal area and proceed with the above method for the line segment equeling to 16/113
2. Extend BA to H such that AH = 4 CO
3. Draw a semi circle on HG.
4. Locate J on ~~A~~HG such that HJ = CO and draw KJ perpendicular to HG.
5. Construct the square JKLM.

Now the area of JKLM = Area of the circle with centre O.

$JK^2 = HJ.JG$

$JK^2 = JG.HJ^2$ ( since HJ = 1)

$\therefore JK^2 = \pi r^2$

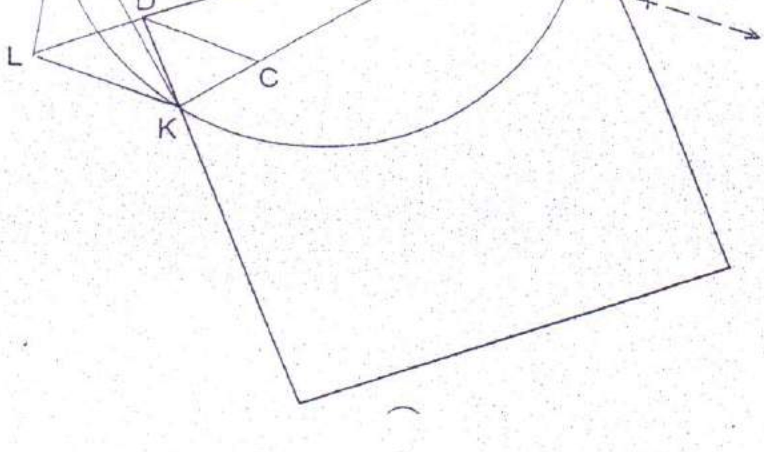
Remark

Though this method is simple it doesn't provide scope to trace back the circle. Also larger the circle we have to covert into square, HG (Fig 3) becomes more and more larger. (It might be the reason, for Srinivasa Ramanujan chose another brilliant method of construction accommodating the entire process in the same circle considered for the conversion into a square! with the same value for π i.e., 355/113)

**SRINIVASA RAMANUJAN'S METHOD OF SQUARING A CIRCLE :**

Ramanujan's mode of solution for the problem, displaying his unmatchable ingenuity, goes as follows :

FIG. 4



**Construction** :

1. Let O be the center and PR any diameter of the given circle.
2. Bisect OP at H and trisect OR at T (2 : 1)
3. Draw TQ perpendicular to OP.
4. Draw RS = TQ
5. Join PS.
6. Draw OM and TN parallel to RS.
7. Draw PK = PM, and PL = MN and perpendicular to OP.
8. Join RL, RK and KL.
9. Cut off RC = RH.
10. Draw CD parallel to KL.

Now $RD^2 = \odot O$.

[On calculation $\pi = \frac{RD^2}{OR^2} = \frac{355}{113}$ . For proof see Appendix – IV]

Ramanujan, it seems, did not try the reversal, i.e., circling a square.

From these solutions given by Bodhayana, Jakob de Galder and Srinivasa Ramanujan, we can conclude that :

1. In order to arrive at different values for the transcendental number $\pi$ through the equation $\frac{x^2}{r^2} = \pi$, where x is the side of the square and r, the radius of the circle, Bodhayana, Srinivasa Ramanujan and Jakob de Galder adopted different modes of constructions.

**Q.No.1 Can't we have a fixed single method for obtaining any plausible value for $\pi$ ?**

2. Another noteworthy point is that their non-reversible nature, i.e., tracing back the circle from the square or vice versa, seems to be far from practicability.

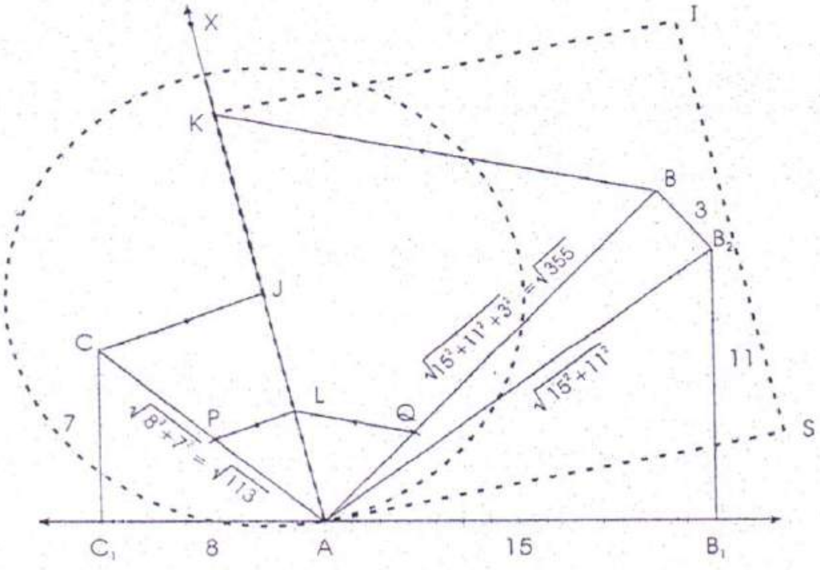
**Q.No.2 Can't we have a method which can provide a feasibility for reversal ?**

In this background, I suggest a single fixed method fusing the answers for the above self-posed potent questions- Q.N0.1 and Q.No.2 – which are the objectives of this article, namely.

1. Obtaining any nearest value for $\pi$ through, $\frac{x^2}{r^2} = \pi$, x being the side of the square and r, the radius of the circle.
2. Providing the nature of reversibility in construction.

## CONSTRUCTION FOR SQUARING A CIRCLE :

FIG. 5




**Construction :**

1. Using Pythagorean principle construct two line segments $\overline{AB}$ of length $\sqrt{355}$ units $\left[AB = \sqrt{AB_1^2 + B_1B_2^2 + B_2B^2} = \sqrt{15^2+11^2+3^2} = \sqrt{355}\right]$ and $\overline{AC}$ of length $\sqrt{113}$ units $\left[Ac = \sqrt{Ac_1^2 + c_1c_2^2} = \sqrt{8^2+7^2} = \sqrt{113}\right]$ making any convenient angle.
2. Draw any arbitrary angle divider $\overrightarrow{AX}$ of $\angle BAC$.
3. Cut off P and Q on $\overline{AC}$ and $\overline{AB}$ respectively with any convenient length such that $\overline{AP} = \overline{AQ}$ and each less than $\overline{AC}$.
4. Locate J on $\overrightarrow{AX}$ such that AJ equals to the radius of the circle whose area is to be made equal to the area of a square.
5. Join CJ and draw PL parallel to CJ cutting $\overrightarrow{AX}$ at L.
6. Join LQ and draw BK parallel to LQ cutting $\overrightarrow{AX}$ at K.
7. Now AK becomes the side of the square KASI whose area equals to the area of the given circle with J as center and JA as radius.

**Proof** : Since PL || CJ,

$\Delta APL \sim \Delta ACJ$

$$\Rightarrow \frac{AC}{AP} = \frac{AJ}{AL}$$

$$\Rightarrow \frac{\sqrt{113}}{AP} = \frac{AJ}{AL}$$

$$\Rightarrow AJ^2 = 113 . \frac{AL^2}{AP^2} \quad \ldots\ldots\ldots\ldots (1)$$

**Similarly,**

$\because BK \parallel QL,$

$\Delta\, AQL \sim \Delta\, ABK$

$\therefore \frac{AB}{AQ} = \frac{AK}{AL}$

$\Rightarrow \frac{\sqrt{355}}{AQ} = \frac{AK}{AL}$

$\Rightarrow AK^2 = 355. \frac{AL^2}{AQ^2}$ . . . . . . . . . (2)

$\Rightarrow$ On dividing (2) by (1) we get.

$\frac{AK^2}{AJ^2} = 355. \frac{AL^2}{AQ^2} \cdot \frac{1}{113} \cdot \frac{AP^2}{AL^2}$

$\Rightarrow \frac{AK^2}{AJ^2} = \frac{355}{113}$ ( $\because$ by construction $AP^2 = AQ^2$)

$\Rightarrow AK^2 = \frac{355}{113} \cdot AJ^2$

$= \pi r^2$ [ $\because r = AJ$ ] ( Q.E.D )

*Then, what about CIRCLING A SQUARE ?*

**SEE FIG NO.5**

**Construction** :

1. Just follow the steps from step (1) to step (3) of the above construction for squaring a circle.
2. Locate K on $\overrightarrow{AX}$ such that AK equals to the side of the given square KASI whose area is to be made equal to the area of a circle.
3. Join KB and draw QL Parallel to KB cutting $\overrightarrow{AX}$ at L.
4. Join LP and draw CJ parallel to LP cutting $\overrightarrow{AX}$ at J.
5. Describe a circle with J as center and JA as radius.

Now the area of the circle equals to the given circle KASI.

It is evident that previous proof holds good for this construction also.

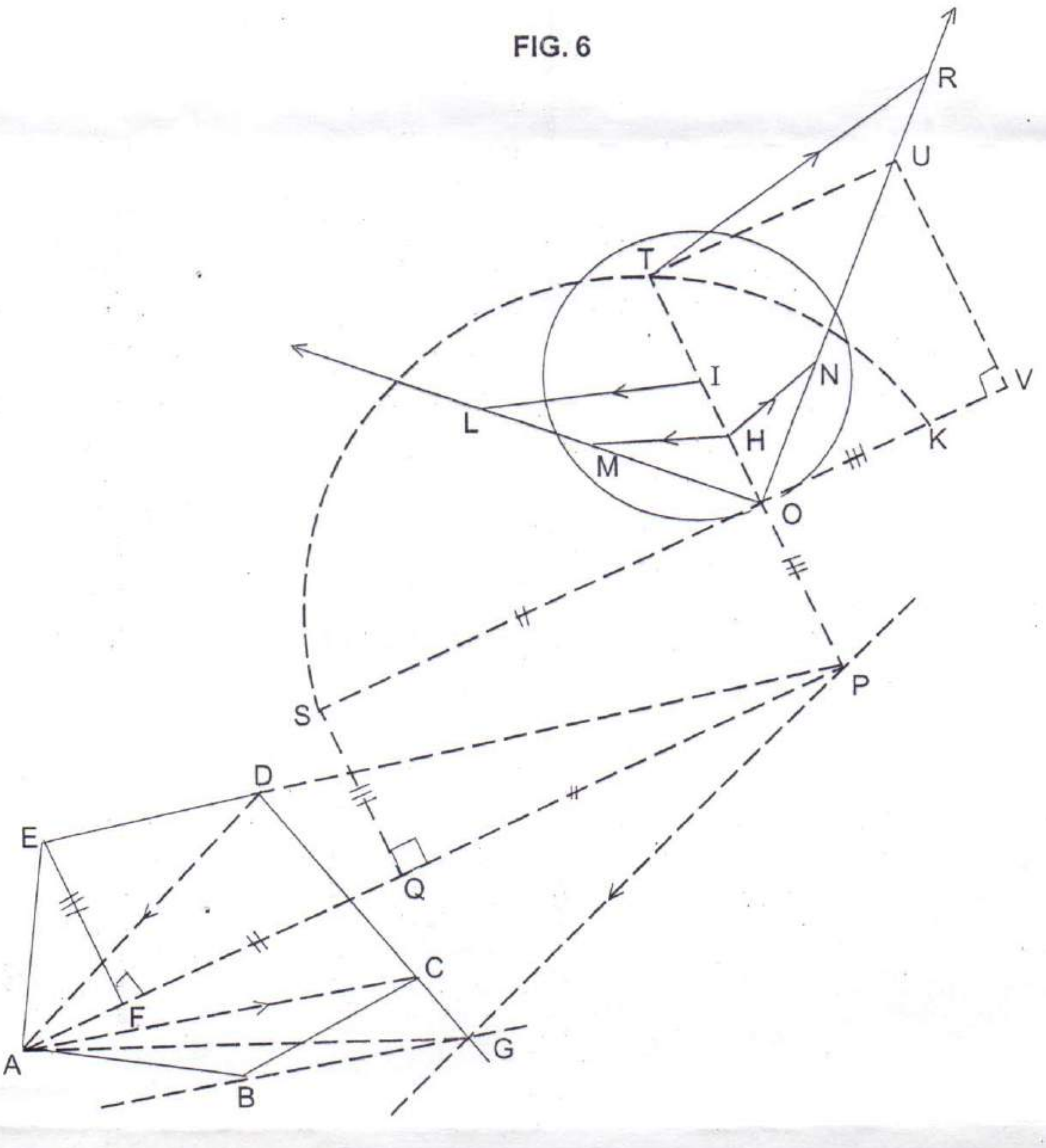
Here I chose the value for $\pi$ as $\frac{355}{113}$ which was given by VIRASENA (Indian) of 480 AD, and TSU – CHANG – CHI (Chinese) because the value, in the words of Srinivasa Ramanujan, appreciably narrows down the difference between the area of the circle and square to $\frac{1}{10,000}$ square inch, if the area of the circle is 14 square miles.

As Pythagorean principle provides a simple way to construct any number of the form $\sqrt{x}$, x ∈ N, we can construct the line segments AB = $\sqrt{a}$ and AC = $\sqrt{b}$, a, b ∈ N, thereby get the nearest plausible rational form for π as $\frac{a}{b}$., like $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, $\frac{103993}{33102}$, $\frac{104348}{33215}$ ....... (from continued fractions, Lambert 1770)

Here, it is not uncommon to hear someone saying that larger the numbers 'a' and 'b' of the rational $\frac{a}{b}$ representing the desired approximate value of π, one has to construct more and more lines to arrive at the line segments representing $\sqrt{a}$ and $\sqrt{b}$. But according to ~~FERMAT~~ Legrange of 17th century Mathematician "Every positive integer is a sum of 1, 2, 3, or 4 squares only"(also called as Four Square Theorem), and he proved it. So for any rational representation of π we need to construct at the most 4 line segments! Therefore using the Pythagorean principle the difficulty in construction of the line segments representing the numbers involved in the plausible rational form of π is nullified.

Another interesting aspect is, that this method of circling a square can be extended to construct a circle equaling the area of any given convex polygon of n sides. It is evident if we have a glance at the Fig No. 6.

**FIG. 6**



Polygon ABCDE = $\Delta$ AED + $\Delta$ ADC + $\Delta$ ABC

= $\Delta$ AED + $\Delta$ ADC + $\Delta$ AGC [$\because \Delta$ ABC = $\Delta$ AGC]

= $\Delta$ AED + $\Delta$ AGD [$\because \Delta$ AGC + $\Delta$ ADC = $\Delta$ AGD]

= $\Delta$ AED + $\Delta$ APD [$\because \Delta$ AGD = $\Delta$ APD]

= $\Delta$ APE [$\because \Delta$ APD + $\Delta$ ADE = $\Delta$ APE]

= $\frac{1}{2}$ . AP . EF

= PQ.QS [$\because$ EF = QS ]

= $\square$ OPQS

= $\square$ OVUT

= $\odot$ I

= The circle with I as centre and OI as radius!

## APPENDIX – I

Fig No.1

Proof :

Let a be the side of the square.

OD = $\frac{1}{2}$ the diagonal of the square.

$$= \frac{a\sqrt{2}}{2}$$

$$= \frac{a}{\sqrt{2}}$$

$$\therefore \quad OF = \frac{a}{2} + \frac{1}{3}\left[\frac{a}{\sqrt{2}} - \frac{a}{2}\right]$$

$$\therefore \ 2\ OF = \frac{a}{3}\ (2 + \sqrt{2})$$

$\therefore \ d = \frac{a}{3}.(2 + \sqrt{2})$, where d is the diameter of the circle.

$$\Rightarrow \quad \frac{a}{d} = \frac{3}{2 + \sqrt{2}}$$

$$\Rightarrow \quad \frac{a}{d} = \frac{3(2 - \sqrt{2})}{2} \ ............. \quad (1)$$

Now $\frac{\pi d^2}{4} = a^2$ gives

$$\frac{a}{d} = \frac{\sqrt{\pi}}{2}$$

$$\therefore \quad \frac{\sqrt{\pi}}{2} = \frac{3}{2}\ (2 - \sqrt{2})$$

$$\Rightarrow \quad \sqrt{\pi} = 3\left[2 - \left(1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34.}\right)\right]$$

$$\because \sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34.}$$

$$\therefore \sqrt{\pi} = \left[\frac{3 \times 239}{12 \times 34}\right]^2$$
$$= \frac{514089}{166464}$$
$$= 3.088$$

## APPENDIX – II

Proof :

If 'a' be the side of the square and 'd' be the diameter of the circle.

$$\therefore \ a = d\left[1 - \frac{1}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8}\right]$$

Here again $\frac{\pi d^2}{4} = a^2$ gives

$$\frac{\sqrt{\pi}}{2} = \frac{a}{d} = \left[1 - \frac{1}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8}\right]$$

$$\therefore \ \sqrt{\pi} = \frac{9785}{5568}$$

$$\text{or } \pi = \frac{95746225}{31002624}$$
$$= 3.088$$

## APPENDIX – III

For Fig No. 2

$$AD^2 = AO^2 + OD^2 = 1 + \frac{49}{64} = \frac{113}{64}$$

In the $\Delta$ DAO, FE || DO

$$\frac{AE}{AO} = \frac{AF}{AD}$$

$$\Rightarrow AE^2 = \frac{AF^2}{AD^2} \times AO^2 = \frac{1/4}{113/64} \times 1 = \frac{16}{113}$$

Similarly in $\Delta$ AED, FG || DE

$$\frac{AG}{AE} = \frac{AF}{AD}$$

$$\Rightarrow AG^2 = \frac{AF^2}{AD^2} \times AE^2 = \frac{16}{113} \times \frac{16}{113} = \frac{16^2}{113^2}$$

$$\therefore \ AG = \frac{16}{113}$$

## APPENDIX – IV

Proof :

Let 'd' and r are the diameter and radius of the circle respectively.

$$RS^2 = TQ^2 = PT.TR = \frac{5}{6} d \ . \ \frac{1}{6} d = \frac{5}{36} d^2$$

So $PS^2 = PR^2 - RS^2$

$$= d^2 - \frac{5}{36} d^2 = \frac{31}{36} d^2$$

Since in $\Delta$ PSR, MO || SR

$$\frac{PM}{PS} = \frac{PO}{PR} = \frac{1}{2} \qquad \left[\because PO = \frac{1}{2} PR\right]$$

$$\therefore \quad PM^2 = \frac{1}{4} PS^2$$

$$= \frac{31}{144} \cdot d^2$$

Also $\frac{MN}{PM} = \frac{OT}{PO} = \frac{\frac{2}{3} \cdot r}{r} = \frac{2}{3}$

$$\therefore \quad MN^2 = \frac{4}{9} PM^2$$

$$\frac{4}{9} \cdot \frac{31}{144} d^2 = \frac{31}{324} \cdot d^2$$

Now $RL^2 = RP^2 + PL^2$

$$= d^2 + MN^2$$

$$= d^2 + \frac{31}{324} \cdot d^2$$

$$= \frac{355}{324} d^2$$

Now $RK^2 = RP^2 - PK^2$

$$= d^2 - PM^2$$

$$= d^2 - \frac{31}{144} d^2$$

$$= \frac{113}{144} d^2$$

Now $\Delta$ RCD ~ $\Delta$ RKL

$$\therefore \quad \frac{RD}{RC} = \frac{RL}{RK}$$

Hence $RD^2 = \frac{RL^2}{RK^2} \cdot RC^2$

$$= \frac{355}{324} \text{ x. } \frac{144}{113} \text{ x } \frac{9}{16} d^2$$

$$= \frac{355}{113} \text{ x. } \frac{d^2}{4}$$

$$\therefore \quad RD^2 = \frac{355}{113} r^2$$

$$= \pi r^2$$

## APPENDIX IV

The relevant verses (about ) are found in the sixth Parva, called Bhismaparva, of the Mahabharata. The lines are

परिमण्डलो महाराज स्वर्भानुः श्रूयते ग्रहः ।
योजनानां सहस्राणि विष्कम्भो द्वादशास्य वै ॥४०॥
परिणाहेन षट्त्रिंशद्विपुलत्वेन चानघ ।
षष्टिमाहुः शतान्यस्य बुधाः पौराणिकास्तथा ॥४१॥
चन्द्रमास्तु सहस्राणि राजन्नेकादश स्मृतः ।
विष्कम्भेण कुरुश्रेष्ठ त्रयस्त्रिंशत्तु मण्डलम् ।
एकोनषष्टिर्वैपुल्याच्छीत रश्मेर्महात्मनः ॥४२॥
सूर्यस्त्वष्टौ सहस्राणि द्वे चान्ये कुरुनन्दन ।
विष्कम्भेण ततो राजन्मण्डलं त्रिंशतं समम् ॥४३॥
अष्टपञ्चाशतं राजन्विपुलत्वेन चानघ ।

These may be translated thus:

'O Great King! It is heard that the Rabu Graha is parimandala (round on all sides), and its diameter is twelve thousand yojanas. (40)

O Sinless King, in circumference, it is thirty six (thousand yojanas), and its thickness is said to be sixty hundred (yojanas) by experts in Puranas. (41)

O King, the Moon is described to be eleven thousand (yojanas) in diameter. O kurusrestha, the peripheral circle thereto is thirty three (thousand yojanas). O Great Soul, the Moon is fifty nine (hundred yojanas) in thickness. (42)

O Kurunandana, the Sun in eight thousand plus another two (thousand yojanas) in diameter. O King, the peripheral circle therefrom is equal to thirty (thousand yojanas). (43)

O Sinless King, it is fifty eight hundred (yojanas) in thickness.

The dimensions described above may be tabulated as follows (all figures are in yojanas):

| Graha | Diameter, D | Circumference, C | Thickness, t |
|---|---|---|---|
| Rahu | 12000 | 36000 | 6000 |
| Moon | 11000 | 33000 | 5900 |
| Sun | 10000 | 30000 | 5800 |

Thus the C/D in all the three cases is 3 which, therefore, is the (approximate) value of $\pi$ used without any doubt.

The same value can be drawn from the verses in The Bible (I Kings 7:23). The verses runs as follows:

*'There He made the sea of cast bronze,* **ten cubits** *from one brim to the other; it was completely round. It's height was five cubits and a line of* **thirty cubits** *measured its circumference.'*

### Books referred :


1. *"Ancient Indian Mathematics and Vedha" by L.V. GURJAR, M.A., Ideal Book Service – Poona –2*
2. *"An introduction to creativity of Ramanujan" by P.K. Srinivasan – Published by The Association of Mathematics Teachers of India – Madras.*
3. *" Topics in Algebra" by I.N. Herstein.*
4. *"$\pi$, A Source Book" By L. Berggren, J. Borwein and Peter Borwein Published by Springer*
5. *"Continued Fractions" by C.D.Olds, The L.W.Singer Company.*
6. *"The value of $\pi$ in The Mahabharatha" by R.C. Gupta, Vol - 12, Nos. 1-2 (1990) pp 45-47 of Ganitha Bharati.*
7. *The Bible.*